A New Look at Newton-Cartan Gravity

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We give a short overview of Newton-Cartan geometry and gravity including its matter couplings. We also present results on a new non-relativistic gravity model in three spacetime dimensions, called extended Bargmann gravity, and show that this model has matter couplings that differ from those of Newton-Cartan gravity.

Keywords: Newton-Cartan gravity, non-relativistic field theories.

1. Introduction

It is well known that in Newtonian gravity free-falling frames are connected by the Galilean symmetries which consist of (constant) time translations, spatial translations, spatial rotations and Galilean boosts. These Galilean boosts rotate space into time but not the other way around since Newtonian time is absolute. In such free-falling frames one does not experience any gravitational force. Such a force is only felt in frames that are not free-falling. For instance, in a (non-rotating) earth-based frame, that is in constant acceleration with respect to a free-falling frame, one experiences a gravitational force that is described by the Newton potential satisfying a Poisson equation. Of course, in a different frame than an earth-based frame, one experiences a different gravitational force that, in principle, can be calculated by relating that frame to a given earth-based or free-falling frame. However, a truly frame-independent formulation of Newtonian gravity was never given by Newton and his followers. The reason for this is that in order to give such a frame-independent formulation one needs a piece of mathematics that was not yet developed around that time. It was only in the middle of the 19th century that Riemann developed the required tool that is now called Riemannian geometry.

When Einstein invented his General Relativity theory in 1915 he achieved two things. First of all, he gave a description of gravity that is consistent with the theory of Special Relativity that he had developed 10 years earlier, by relating inertial frames to each other via the Poincaré transformations and by making use of the geometry of spacetime to give a proper description of gravity. In this way he built in a delay effect that avoided the instantaneous action of Newton's gravitational force. But most importantly, and this took Einstein many years of hard work to achieve, he presented his equations in a frame-independent way. For this, he needed the Riemannian geometry mentioned above and communicated to him by his friend Albert Grossmann. The Poincaré symmetries differ from the Galilean symmetries only as far as the boosts are concerned. Unlike the Galilean boosts the Lorentzian boosts rotate space into time *and* time into space: the concept of time is relative in Einstein's theory. Furthermore, to obtain a frame-independent formulation Einstein introduced a symmetric tensor field to describe the gravitational force. This field replaces the Newton potential and describes geometrical distances in the Riemannian spacetime manifold.

It was only 8 years later that Élie Cartan did for Newtonian gravity what Einstein had achieved for relativistic gravity. The formulation of Newtonian gravity in an arbitrary frame goes under the name of Newton-Cartan (NC) gravity. This NC gravity theory contains more fields that just the Newton potential. Locally, the formulation given by Newton, with a Newton potential in an earth-based frame, can easily be obtained from the general formulation by an appropriate gauge-fixing of the gravitational fields such that one is left with the Newton potential as the only non-zero field. The geometry Cartan was using is called NC geometry. This NC geometry differs from the Riemannian geometry used by Einstein in the sense that it has a *degenerate* metric and a *unique* foliation with an absolute time direction.

Given NC geometry and gravity the question arises: why should we study non-relativistic gravity? There are two main reasons why NC gravity has received a new appreciation in recent years. First of all, it arises in the context of the holographic principle which states that all the information about a gravitational theory in a given volume can be encoded by a different non-gravitational quantum field theory that lives on the surface surrounding this volume. This holographic principle has found a precise mathematical framework in string theory where it goes under the name of the AdS/CFT correspondence. This is a special situation where the gravity theory lives in a maximally symmetric spacetime with a negative cosmological constant, namely an Anti-de Sitter (AdS) spacetime, and where the field theory is a special so-called conformal field theory (CFT). In recent years much research has been done on non-AdS and non-relativistic holography to understand the validity and the basic principles underlying the holographic principle. One of the simplest deviations of AdS, breaking the relativistic isometries, is a Lifshitz spacetime which has less symmetries than AdS. Correspondingly, it has been found that at the field theory side the relativistic scale invariance of the CFT is broken to a non-relativistic scale invariance corresponding to a field theory that couples to an extension of NC geometry with so-called 'twistless torsion'[1]. Independently, NC geometry has recently found applications in the condensed matter physics community. Here one works with an Effective Field Theory (EFT) coupled to NC geometry to describe non-perturbative features of models such as the fractional quantum Hall effect[2], chiral superfluids and simple fluids. The coupling to NC gravity means that one uses an arbitrary frame formulation in which general features are visible. One can compare this with the Coriolis force that is not visible in a non-rotating earth-based frame but can only be observed in a more general (rotating) frame.

Having the above motivation in mind we will first show in section 2 how NC gravity can be obtained via a kind of gauging procedure from the centrally extended Galilei algebra which is called the Bargmann algebra. In section 3, we will discuss some recent results on matter couplings. Next, in section 4 we discuss NC gravity with torsion while in section 5 we present a new model of gravity in three spacetime dimensions, which we call Extended Bargmann Gravity (EBG). In section 6 we show that this EBG model has matter couplings that differ from the usual matter couplings that occur in NC gravity. Finally, in section 7 we discuss future directions.

2. Newton-Cartan from gauging Bargmann

Let us first remind ourselves how to obtain Einstein gravity via a kind of gauging procedure from the Poincaré algebra. In General Relativity all free-falling frames are connected by the following Poincaré symmetries:

- space-time translations: $\delta x^\mu = \xi^\mu\,,$
- Lorentz transformations: $\delta x^\mu = \lambda^\mu{}_\nu\, x^\nu\,.$

In arbitrary frames the gravitational force is described by the metric field. Instead of a metric, it is convenient to use an equivalent Vierbein formulation, with Vierbein field $E_\mu{}^A$ ($\mu = 0,1,2,3; A = 0,1,2,3$) since these Vierbeine are naturally related to the gauge fields of Poincaré translations.

In the non-relativistic case all free-falling frames are connected by the Galilean symmetries:

- time translations: $\delta t = \xi^0\,,$
- space translations: $\delta x^i = \xi^i\,,\qquad i = 1,2,3\,,$
- spatial rotations: $\delta x^i = \lambda^i{}_j\, x^j\,,$
- Galilean boosts: $\delta x^i = \lambda^i t\,.$

They are identical to the Poincaré symmetries except for the Galilean boosts which differ from the Lorentzian boosts as we discussed in the Introduction.

It is important to distinguish Newtonian gravity from Newton-Cartan gravity. Newtonian gravity is valid in frames of constant acceleration with respect to free-falling frames and is described by a single Newton potential $\Phi(x)$. On the other hand, Newton-Cartan gravity is valid in arbitrary frames and needs more fields to describe the gravitational force. To be precise, the required fields are a so-called temporal Vierbein $\tau_\mu(x)$ and a spatial Vierbein $e_\mu{}^a(x)$. Since these two fields together form a 4×4 matrix $\{\tau_\mu, e_\mu{}^a\}$ one would think that these fields suffice. Surprisingly, one needs one more field to describe Newton-Cartan gravity, namely a vector field $m_\mu(x)$.

One way to understand why this extra field is needed is to compare a freely moving relativistic particle with its non-relativistic counterpart. On the one hand a relativistic particle is described by the action

$$S_{\text{relativistic}} = -m \int d\tau \, \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \qquad \mu = 0, 1, 2, 3\,, \tag{1}$$

where $x^\mu(\tau)$ are the embedding coordinates. Clearly, the Lagrangian corresponding to this action is invariant under the Poincaré symmetries. On the other hand, a non-relativistic particle is described by the action

$$S_{\text{non-relativistic}} = \frac{m}{2} \int \frac{\dot{x}^i \dot{x}^j \delta_{ij}}{\dot{t}} d\tau \qquad i = 1, 2, 3\,. \tag{2}$$

In this case the Lagrangian is <u>not</u> invariant under Galilean boosts. Instead, the Lagrangian transforms with a total derivative as follows:

$$\delta L_{\text{ non-relativistic}} = \frac{d}{d\tau}(m x^i \lambda^j \, \delta_{ij})\,. \tag{3}$$

Although the action is invariant, the non-invariance of the Lagrangian leads to modified Noether charges which induce a central extension of the underlying Galilei algebra. One thus ends up with the Bargmann algebra where the gauge field of the extra central charge transformation is the vector field $m_\mu(x)$.

Before gauging the Bargmann algebra it is of interest to compare gaugings and Inönü-Wigner contractions of algebras and taking the non-relativistic limit of gravity. We have indicated the relations between these different manipulations below.

$$\begin{array}{ccc}
\text{Poincaré} \otimes \text{U}(1) & \overset{\text{`gauging'}}{\Longrightarrow} & \text{General Relativity} \otimes \text{U}(1) \\
\text{contraction} \ \Downarrow & & \Downarrow \ \text{non-relativistic limit} \\
\text{Bargmann} & \overset{\text{`gauging'}}{\Longrightarrow} & \text{Newton-Cartan gravity}
\end{array}$$

Fig 1. This Figure indicates the different relations between gaugings, contractions and non-relativistic limits.

We see that, in order to obtain the Bargmann algebra from a contraction of the Poincaré algebra we need first to extend the Poincaré algebra with an additional U(1) generator, in order to account for the central charge generator which is present in the Bargmann algebra on top of the usual Galilei generators. This suggests that the non-relativistic limit of General Relativity can only be taken in the presence of an additional vector field that corresponds to the extra U(1) generator. This non-relativistic limit should mimic the Inönü-Wigner contraction of the algebra.

We now return to the gauging of the Bargman algebra[3] which is based on a similar gauging procedure developed in the supergravity community many years ago [4]. Our starting point is the set of commutation relations defining the Bargmann algebra

$$
\begin{aligned}
&[J_{ab}, P_c] = -2\delta_{c[a}P_{b]}\,, && [J_{ab}, G_c] = -2\delta_{c[a}G_{b]}\,, \\
&[G_a, H] = -P_a\,, && [G_a, P_b] = -\delta_{ab}Z\,, \qquad a = 1, 2, \ldots, d\,,
\end{aligned} \tag{4}
$$

where $\{H, P_a, J_{ab}, G_a, Z\}$ are the generators of time translations, space translations, spatial rotations, Galilean boosts and central charge transformations, respectively. In this gauging procedure we associate to every generator/symmetry a gauge field, gauge parameters that are arbitrary functions of spacetime and covariant curvatures as indicated in the table below.

symmetry	generators	gauge field	parameters	curvatures
time translations	H	τ_μ	$\zeta(x^\nu)$	$\mathcal{R}_{\mu\nu}(H)$
space translations	P^a	${e_\mu}^a$	$\zeta^a(x^\nu)$	${\mathcal{R}_{\mu\nu}}^a(P)$
Galilean boosts	G^a	${\omega_\mu}^a$	$\lambda^a(x^\nu)$	${\mathcal{R}_{\mu\nu}}^a(G)$
spatial rotations	J^{ab}	${\omega_\mu}^{ab}$	$\lambda^{ab}(x^\nu)$	${\mathcal{R}_{\mu\nu}}^{ab}(J)$
central charge transf.	Z	m_μ	$\sigma(x^\nu)$	$\mathcal{R}_{\mu\nu}(Z)$

Table 1. This table indicates for every symmetry the corresponding generators, gauge fields, local gauge parameters and covariant curvatures.

From the Table we see that besides a timelike Vierbein τ_μ and a spatial Vierbein ${e_\mu}^a$ [a] there are two independent spin-connection fields $\{{\omega_\mu}^{ab}, {\omega_\mu}^a\}$ of spatial rotations and Galilean boosts, respectively, and a gauge field m_μ for the central charge transformations. Following General Relativity, in order to make the spin-connection fields dependent we need to impose constraints on the curvatures. Unlike General Relativity, the curvature $\mathcal{R}_{\mu\nu}(H)$ of time translations cannot play any role here since that curvature does not contain any of the two spin-connections fields. At this point the curvature $\mathcal{R}_{\mu\nu}(Z)$ of the central charge transformations comes to help since that curvature does contain the spin-connection field of the Galilean boosts. Independent of this we do set the curvature of time translations to zero since this defines the foliation of spacetime. We thus arrive at the following set of curvature

[a] Even though these Vielbeine are not invertible, one can define projective inverses τ^μ, ${e^\mu}_a$ via the relations $\tau^\mu\tau_\mu = 1$, $\tau^\mu {e_\mu}^a = 0$, $\tau_\mu {e^\mu}_a = 0$, ${e^\mu}_a {e_\mu}^b = \delta^b_a$, ${e^\mu}_a {e_\nu}^a = \delta^\mu_\nu - \tau^\mu\tau_\nu$.

constraints:

$$\mathcal{R}_{\mu\nu}{}^{a}(P) = 0\,, \qquad \mathcal{R}_{\mu\nu}(Z) = 0\;: \qquad \text{solve for spin-connection fields} \tag{5}$$

$$\mathcal{R}_{\mu\nu}(H) = \partial_{[\mu}\tau_{\nu]} = 0 \;\to\; \tau_\mu = \partial_\mu \tau\;: \quad \text{absolute time ('zero torsion')} \tag{6}$$

$$\mathcal{R}_{\mu\nu}{}^{ab}(J) \neq 0\;: \qquad \text{un-constrained off-shell} \tag{7}$$

$$\mathcal{R}_{0(a,b)}(G) \neq 0\;: \qquad \text{un-constrained off-shell} \tag{8}$$

Note that the zero torsion constraint (7) allows us to solve for the timelike Vierbein in terms of an arbitrary function $\tau(x^\nu)$ of the spacetime coordinates. Choosing $\tau(x^\mu) = t$ defines the time-coordinate t to be the absolute time but there are other choices possible as well.

Following the standard gauging procedure one ends up with three independent gauge-fields $\{\tau_\mu, e_\mu{}^a, m_\mu\}$ that transform under general coordinate transformations, with parameters ξ^μ, as covariant vectors and under the other Bargmann symmetries as follows:

$$\begin{aligned}
\delta\tau_\mu &= \xi^\lambda\partial_\lambda\tau_\mu + \partial_\mu\xi^\lambda\tau_\lambda\,,\\
\delta e_\mu{}^a &= \xi^\lambda\partial_\lambda e_\mu{}^a + \partial_\mu\xi^\lambda e_\lambda{}^a + \lambda^a{}_b\, e_\mu{}^b + \lambda^a\tau_\mu\,,\\
\delta m_\mu &= \xi^\lambda\partial_\lambda m_\mu + \partial_\mu\xi^\lambda m_\lambda + \partial_\mu\sigma + \lambda_a\, e_\mu{}^a\,.
\end{aligned} \tag{9}$$

Furthermore, one may define two Galilean-invariant metrics

$$\tau_{\mu\nu} = \tau_\mu\tau_\nu\,, \qquad\qquad h^{\mu\nu} = e^\mu{}_a e^\nu{}_b\,\delta^{ab}\,,$$

one in the time direction and a separate one in the spatial directions. Note that the timelike metric is only defined with lower indices whereas the spatial metric is only defined with upper indices. Without the central charge vector field it is not possible to define a timelike metric with upper indices and a spatial metric with lower indices that is invariant under Galilean boosts. Such unwanted variations can only be canceled by adding m_μ-dependent terms to these metrics.

Now that we have defined the symmetries of NC gravity in arbitrary frames it is easy to switch between frames. For instance, to go from the general frame formulation back to the free-falling frames only, one must impose the following gauge-fixing conditions eliminating, locally, all gravitational fields:

$$\tau_\mu = \delta^t_\mu\,, \qquad e_t{}^a = 0, \quad e_i{}^a = \delta^a_i\,, \qquad m_\mu = 0\,. \tag{10}$$

This leads to the following non-relativistic Killing equations:

$$\begin{aligned}
&\partial_\mu\xi^t = 0\,, \qquad \partial_t\xi^i + \lambda^i = 0\,,\\
&\partial_i\xi^j + \lambda^j{}_i = 0\,, \qquad \partial_t\sigma = 0\,, \qquad \partial_i\sigma + \lambda_i = 0\,,
\end{aligned} \tag{11}$$

whose most general solution is given by the Galilean symmetries connecting free-falling frames:

$$\xi^t(x^\mu) = \zeta\,, \qquad \xi^i(x^\mu) = \xi^i - \lambda^i\, t - \lambda^i{}_j\, x^j\,, \qquad \sigma(x^\mu) = \sigma - \lambda^i\, x^i\,. \tag{12}$$

Instead, one could also go from general frames to frames with constant acceleration. In that case one has to impose less stringent gauge-fixing conditions in which the Newton potential survives as one of the components of the gravitational fields. This gauge-fixing automatically gives the correct transformation rule of the Newton potential under the Bargmann symmetries.

So far, we have only defined the kinematics of NC gravity. To define the dynamics we need to impose equations of motion. For this purpose we introduce the following equations:

$$\tau^\mu e^\nu{}_a \mathcal{R}_{\mu\nu}{}^a(G) = 0 \qquad\qquad \mathbf{1} \tag{13}$$

$$e^\nu{}_a \mathcal{R}_{\mu\nu}{}^{ab}(J) = 0 \qquad\qquad \mathbf{a} + (\mathbf{ab})\,, \tag{14}$$

where we have indicated at the right the representations of spatial rotations to which these equations belong. The first singlet equation reduces to the Poisson equation for the Newton potential after gauge-fixing to frames with constant acceleration. Note that, without the second equation, the first equation would not be invariant under Galilean boosts. The number of equations is the same as in General Relativity but the number of the independent fields is not the same. Therefore, there is no obvious way to integrate the above NC equations of motion to an action.

3. Matter Coupled Newton-Cartan Gravity

One way to add matter to NC gravity is to start from the relativistic answer and take the non-relativistic limit. In this way one obtains matter couplings from arbitrary contracting backgrounds[5].[b] As a bonus this also gives an elegant way to derive non-relativistic field theories from relativistic ones. In the Figure below we have indicated how this works for Klein-Gordon versus Schrödinger.

$$\begin{tikzcd}
\text{scalar + GR} \arrow[r, Rightarrow, "\text{‘limit’}"] & \text{Schrödinger + NC} \arrow[d, Rightarrow, "\text{free-falling frames}"] \\
\text{Klein-Gordon} \arrow[u, Rightarrow, "\text{general frames}"] \arrow[r, Rightarrow, "?"] & \text{Schrödinger}
\end{tikzcd}$$

Fig 2. This Figure indicates how to obtain Schrödinger coupled to NC gravity from Klein-Gordon coupled to General Relativity by taking a non-relativistic limit. It also indicates how, as a bonus, we can obtain pure Schrödinger from pure Klein-Gordon by switching between general and free-falling frames.

We first define the non-relativistic limit of General Relativity without matter by mimicking the Inönü-Wigner contraction of the corresponding algebra as much as possible. This contraction works as follows[6]. Our starting point is the Poincaré

[b]For another recent and related discussion, see[7].

algebra plus an additional U(1) generator $\mathcal{Z}$ that commutes with all the Poincaré generators:

$$\big[P_A, M_{BC}\big] = 2\,\eta_{A[B}\,P_{C]}\,, \quad \big[M_{AB}, M_{CD}\big] = 4\,\eta_{[A[C}\,M_{D]B]} \quad \text{plus} \quad \mathcal{Z}\,. \tag{15}$$

Here $\{P_A, M_{AB}\}$ are the generators of spacetime translations and Lorentz generators, respectively. Next, we decompose $A = (0, a)$ and relate the Poincaré $\otimes$ U(1) generators $\{P_0, P_a, M_{a0}, M_{ab}, \mathcal{Z}\}$ to the Bargmann generators $\{H, P_a, G_a, J_{ab}, Z\}$ as follows:

$$P_0 = \frac{1}{2\omega}\,H + \omega\,Z\,, \qquad \mathcal{Z} = \frac{1}{2\omega}\,H - \omega\,Z\,, \qquad A = (0, a)\,, \tag{16}$$

$$P_a = P_a\,, \qquad M_{ab} = J_{ab}\,, \qquad M_{a0} = \omega\,G_a\,, \tag{17}$$

where we have introduced a contraction parameter ω. In a second step, taking the limit $\omega \to \infty$, we obtain the Bargmann algebra including the following commutator containing the central charge generator Z:

$$\big[P_a, G_b\big] = \delta_{ab}\,Z\,. \tag{18}$$

Inspired by the above Inönü-Wigner contraction we now define the non-relativistic limit of General Relativity as follows. We first introduce, on top of the Vierbein field, a vector field M_μ with $\partial_{[\mu}M_{\nu]} = 0$. Next, we relate the relativistic gauge fields $\{E_\mu{}^A, M_\mu\}$ to the non-relativistic gauge fields $\{\tau_\mu, e_\mu{}^a, m_\mu\}$ as follows:

$$E_\mu{}^0 = \omega\,\tau_\mu + \frac{1}{2\omega}\,m_\mu\,, \quad M_\mu = \omega\,\tau_\mu - \frac{1}{2\omega}\,m_\mu\,, \quad E_\mu{}^a = e_\mu{}^a\,. \tag{19}$$

This implies for the inverse Vierbein fields the following relation:

$$E^\mu{}_a = e^\mu{}_a - \frac{1}{2\omega^2}\,\tau^\mu e^\rho{}_a m_\rho + \mathcal{O}\left(\omega^{-4}\right) \text{ and similar for } E^\mu{}_0\,. \tag{20}$$

The definitions of the non-relativistic inverse fields $\{\tau^\mu, e^\mu{}_a\}$ we have used here can be found in[3].

In a second step we now take the limit $\omega \to \infty$. In this way we obtain the correct non-relativistic transformation rules (9) and the equations of motion (13). Note that the standard textbooks on General Relativity usually go straight from General Relativity to Newtonian gravity skipping the general frame formulation of NC gravity.

As an example we consider a complex scalar of mass M with Lagrangian given by

$$E^{-1}\,\mathcal{L}_{\rm rel} = -\frac{1}{2}\,g^{\mu\nu}\,D_\mu\Phi^* D_\nu\Phi - \frac{M^2}{2}\,\Phi^*\Phi \quad \text{with} \tag{21}$$

$$D_\mu\Phi = \partial_\mu\Phi - \mathrm{i}\,M\,M_\mu\,\Phi\,, \qquad \delta\Phi = \mathrm{i}\,M\,\Lambda\,\Phi\,. \tag{22}$$

In a free-falling frame this Lagrangian reduces to the standard Klein-Gordon Lagrangian. Note that M_μ is not an electromagnetic gauge field. The mass M is not equal to the electric charge q. Instead the gauge field M_μ couples to the current expressing the conservation of # particles - # anti-particles.

We now take the non-relativistic limit of General Relativity as defined above together with $M = \omega m\,, \Phi \to \sqrt{\frac{\omega}{m}}\phi$. This leads us to the following Schrödinger Lagrangian coupled to NC gravity:

$$e^{-1}\mathcal{L}_{\text{Schrödinger}} = \Big[\,\frac{\mathrm{i}}{2}\Big(\Phi^*\tilde{D}_0\Phi - \Phi\tilde{D}_0\Phi^*\Big) - \frac{1}{2m}\big|\tilde{D}_a\Phi\big|^2\,\Big] \quad \text{with} \tag{23}$$

$$\tilde{D}_\mu\Phi = \partial_\mu\Phi + \mathrm{i}\,m\,m_\mu\,\Phi\,, \qquad \delta\Phi = \xi^\mu\partial_\mu\Phi - \mathrm{i}\,m\,\sigma\,\Phi\,. \tag{24}$$

In a free-falling frame this is the standard Schrödinger Lagrangian. Note that the non-relativistic gauge field m_μ couples to the current that expresses the conservation of # particles only. Intuitively, the extra vector gauge field takes care of the infinities that occur if you switch between a Lagrangian with 2 time derivatives and a Lagrangian with one time derivative.

4. Newton-Cartan Gravity with Torsion

When studying non-relativistic holography one of the simplest deviations from AdS spacetime to consider is a Lifshitz spacetime which has non-relativistic Lifshitz isometries[8]. The Lifshitz spacetime metric is given by

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{1}{r^2}\big(dr^2 + d\vec{x}^2\big)\,.$$

Here $z \neq 1$ is the dynamical exponent that characterizes the anisotropic scaling

$$t \to \lambda^z t \quad \text{and} \quad \vec{x} \to \lambda\vec{x}\,.$$

The special thing about this Lifshitz spacetime is that when one approaches the boundary ($r \to \infty$) the relativistic lightcone flattens out and becomes, assuming that $z > 1$, a Galilean lightcone[1]. This is the reason that one finds at the boundary a CFT coupled to a NC background geometry. In fact, one even finds an extended NC geometry with a so-called twistless torsion[1].

To clarify this extended NC geometry it is convenient to first adapt the so-called conformal method[9] to the non-relativistic case. The relativistic conformal group is the Poincaré group extended with dilatations D and special conformal transformations K_μ:

$$\text{Conformal} = \text{Poincaré} + D\ (\text{dilatations}) + K_\mu\ (\text{special conf. transf.})\,.$$

The (kinematics of the) corresponding conformal gravity theory can be obtained by gauging the conformal algebra in the same way that the kinematics of General Relativity can be obtained by gauging the Poincaré algebra. The gauge fields b_μ of dilatations and ${f_\mu}^a$ of special conformal transformations are special in the sense that b_μ transforms as a shift under the special conformal transformations with parameter

$\Lambda_K^a(x)$ and $f_\mu{}^a$ is a dependent gauge-field that can be expressed in terms of the independent gauge fields $e_\mu{}^a, b_\mu$ and the dependent spin-connection field $\omega_\mu{}^{ab}(e,b)$:

$$\delta b_\mu = \Lambda_K^a(x) e_\mu{}^a \,, \qquad f_\mu{}^a = f_\mu{}^a(e,\omega,b)\,. \tag{25}$$

It is well-known that in the relativistic case there is a relationship between a Poincaré invariant and the CFT of a real scalar. This relationship goes both ways which can be best explained at the hand of the specific example of the Einstein-Hilbert action. Starting from the D-dimensional Poincaré invariant

$$e^{-1}\mathcal{L} = \frac{1}{\kappa^2}\,R \tag{26}$$

one can 'zip it' to a CFT of a real scalar in two steps. In a first step one writes the Poincaré Vielbein $(e_\mu{}^{\mathrm{A}})^{\mathrm{P}}$ as the product of a compensating scalar φ and a conformal Vielbein $(e_\mu{}^{\mathrm{A}})^{\mathrm{C}}$ as follows:

$$(e_\mu{}^{\mathrm{A}})^{\mathrm{P}} = \kappa^{\frac{2}{D-2}}\,\varphi\,(e_\mu{}^{\mathrm{A}})^{\mathrm{C}}\,. \tag{27}$$

The dilatations of the compensating scalar φ cancel the dilatations of the conformal Vielbein such that the left-hand-side of the above equation is invariant under dilatations. Next, in a second step one fixes the general coordinate transformations and local dilatations to rigid conformal transformations by imposing the following gauge-fixing condition:

$$(e_\mu{}^{\mathrm{A}})^{\mathrm{C}} = \delta_\mu{}^{\mathrm{A}}\,. \tag{28}$$

Upon making the redefinition (assuming $D > 2$)

$$\varphi = \phi^{\frac{2}{D-2}} \tag{29}$$

one finally obtains the following CFT for the real scalar ϕ:

$$\text{CFT}: \quad \mathcal{L} = 4\,\tfrac{D-1}{D-2}\,\phi\Box\phi\,. \tag{30}$$

The above procedure also works the other way around. Starting from the CFT (30) one can 'unzip' it by two steps. First we couple the CFT to conformal gravity by replacing the derivatives by conformal-covariant derivatives:

$$e^{-1}\mathcal{L} = 4\,\tfrac{D-1}{D-2}\,\phi\Box^{\mathrm{C}}\phi\,. \tag{31}$$

In a second step one fixes the local dilatations by imposing the gauge-fixing condition

$$\phi = \frac{1}{\kappa}\,, \tag{32}$$

where κ is the gravitational coupling constant. Substituting this gauge condition back into the Lagrangian (31) one re-obtains the Poincaré invariant (26).

We wish to extend the conformal method to the non-relativistic case. However, we will not use the non-relativistic contraction of the conformal algebra which is the Galilean Conformal Algebra (GCA). The reason for this is that the GCA does not allow for a central extension that plays such an important role in the case of the Bargmann algebra. Instead, we will use the so-called Schrödinger algebra which is somewhat smaller than the GCA but does allow a central extension. It is an extension of the Bargmann algebra that is characterized by a dynamical exponent z. For instance, we have the following commutators

$$[H, D] = zH\,, \qquad\qquad [P_a, D] = P_a\,, \tag{33}$$

where D is the generator of dilatations, H of time translations and P_a of space translations. For $z = 2$ the Schrödinger algebra is the Bargmann algebra extended with dilatations D and a single special conformal transformation K:

$$z = 2 \text{ Schrödinger} = \text{Bargmann} + D \text{ (dilatations)} + K \text{ (special conf.)}\,.$$

For $z = 1$ we obtain the relativistic conformal algebra and for $z \neq 2$ there are no special conformal transformations.

The role of conformal gravity in the relativistic case is taken over by the so-called Schrödinger gravity theory which can be obtained by gauging the $z = 2$ Schrödinger algebra. We find that the independent gauge fields $\{\tau_\mu, e_\mu{}^a, m_\mu\}$ transform as follows:

$$\begin{aligned}
\delta\tau_\mu &= 2\Lambda_D\tau_\mu\,,\\
\delta e_\mu{}^a &= \Lambda^a{}_b\, e_\mu{}^b + \Lambda^a\tau_\mu + \Lambda_D e_\mu{}^a\,,\\
\delta m_\mu &= \partial_\mu\sigma + \Lambda_a\, e_\mu{}^a\,.
\end{aligned}$$

The time projection $\tau^\mu b_\mu$ of b_μ transforms under K as a a shift while the spatial projection $b_a \equiv e_a{}^\mu b_\mu$ is dependent:

$$b_a\,(e,\tau) = e_a{}^\mu \tau^\nu \partial_{[\mu}\tau_{\nu]}\,. \tag{34}$$

This expression is the solution of the following twistless torsionless or modified foliation condition

$$\partial_{[\mu}\tau_{\nu]} - 2b_{[\mu}\tau_{\nu]} = 0\,. \tag{35}$$

We are now able to apply the Schrödinger method, i.e. the non-relativistic analog of the conformal method. The simplest case to consider is to 'unzip' the Schrödinger action for a complex scalar Ψ in d spatial dimensions with dilatation weight $w = -d/2$ and central charge weight M, i.e.

$$\delta\Psi = \big(-\tfrac{d}{2}\lambda_D + iM\sigma\big)\Psi\,. \tag{36}$$

The Schrödinger Field Theory (SFT) for such a complex scalar is given by the Schrödinger action

$$\text{SFT}: \quad S_{\text{ Schrödinger}} = \int dt d^d x\, \Psi^\star\Big(i\partial_0 - \tfrac{1}{2M}\partial_a\partial_a\Big)\Psi\,. \tag{37}$$

Unzipping the Schrödinger action leads to a Galilean invariant that has inconsistent equations of motion by itself. This situation can be compared with the cosmological constant in the relativistic case. As a separate Poincaré invariant the cosmological constant has inconsistent equations of motion but it can perfectly be added to the Einstein-Hilbert action. The same applies to the Galilean invariant that corresponds to the Schrödinger action. This invariant, whose explicit expression we refrain from giving here, can be added to a higher-derivative action of the Hořava-Lifshitz type [1].

Instead of considering these higher-derivative invariants, we will now show that the Schrödinger method can also be applied at the level of the equations of motion. To be specific, we will apply it to the NC equations of motion that cannot be integrated to an action and we will show that the Schrödinger procedure naturally gives rise to the equations of motion of NC gravity with twistless torsion. It is convenient to first consider the case of zero torsion, i.e. $b_a = 0$. The foliation constraint and the equations of motion for this Galilean-invariant case are given by (the index G denotes that we are dealing with the Galilean and not the Schrödinger case)

$$\text{foliation constraint}: \quad \partial_\mu(\tau_\nu)^{\rm G} - \partial_\nu(\tau_\mu)^{\rm G} = 0\,, \tag{38}$$

$$\text{E.O.M.}: \quad (\tau^\mu)^{\rm G}(e^\nu{}_a)^{\rm G}\mathcal{R}_{\mu\nu}{}^a(G) = 0\,, \tag{39}$$

$$(e^\nu{}_a)^{\rm G}\mathcal{R}_{\mu\nu}{}^{ab}(J) = 0\,. \tag{40}$$

Applying the Schrödinger method discussed above we now zip these equations to the following SFT of a complex scalar $\Psi = \varphi e^{i\chi}$:

$$\text{SFT}: \quad \partial_0\partial_0\varphi = 0 \qquad \text{and} \qquad \partial_a\varphi = 0 \quad \text{with} \quad w = 1\,. \tag{41}$$

Note that the SFT is defined in terms of equations of motion only and that the scalar χ is absent implying that we do not break the central charge transformations. We observe that the first equation is not invariant under the Schrödinger transformations by itself but that it transforms into the constraint $\partial_a\varphi = 0$. This constraint is the zipped version of the foliation constraint (38).

We now consider the case with twistless torsion, i.e. $b_a \neq 0$ and start from the SFT side. It is clear that we cannot use the SFT given in (41) since the foliation constraint has been changed and therefore the constraint $\partial_a\varphi = 0$ is no longer valid anymore. In fact, the new foliation constraint, given in eq. (35), is invariant under dilatations and therefore there is no zipped version of it, i.e. there is no constraint. This means that the equation $\partial_0\partial_0\phi = 0$ is not invariant under the Schrödinger transformations. To compensate for this lack of invariance the second scalar χ now comes to our help. One can show that the following modified equation, with extra terms containing the scalar χ defines a modified SFT′:

$$\text{SFT}': \quad \partial_0\partial_0\varphi - \frac{2}{M}(\partial_0\partial_a\varphi)\partial_a\chi + \frac{1}{M^2}(\partial_a\partial_b\varphi)\partial_a\chi\partial_b\chi = 0\,. \tag{42}$$

The advantage of the Schrödinger approach is that it is much easier to find the answer in the zipped SFT version than in the unzipped Galilean invariant version. Nevertheless, the answer in terms of the Galilean invariant equations of motion can be reconstructed by unzipping the SFT′ given above. Following the rules (coupling to Schrödinger gravity and gauge-fixing) one finds the following answer[10]:

$$\boxed{-\triangle\Phi + \hat\tau^\mu\partial_\mu K + K^{ab}K_{ab} - 8\,\Phi\, b\cdot b - 2\,\Phi\,\mathcal{D}\cdot b - 6\, b^a\mathcal{D}_a\Phi = 0\,.} \tag{43}$$

plus

$$\boxed{e^\nu{}_a R_{\mu\nu}{}^{ab}(J) = 0\,.} \tag{44}$$

Truncating to the case of zero torsion one re-obtains the equations of motion of NC gravity (38). This finishes our discussion of NC gravity with twistless torsion.

5. Extended Bargmann Gravity

It is well-known that taking the limit of matter-coupled General Relativity in three spacetime dimensions is special[11]. Consider, for instance, the action of General Relativity plus a complex scalar in D dimensions:

$$S = \int d^Dx\sqrt{-g}\left\{\frac{1}{\kappa^2}R - \frac{1}{2}\,g^{\mu\nu}\,D_\mu\Phi^* D_\nu\Phi - \frac{M^2}{2}\,\Phi^*\Phi\right\}. \tag{45}$$

The equations of motion of the metric tensor corresponding to this action are given by

$$R_{\mu\nu} = \kappa^2\left(T_{\mu\nu} - \frac{1}{D-2}g_{\mu\nu}T\right). \tag{46}$$

Taking the non-relativistic limit in the equations of motion as before and, furthermore, rescaling the gravitational coupling constant κ with

$$\kappa^2 \to \kappa^2/\omega^4 \tag{47}$$

yields the following non-relativistic equations of motion

$$R_{0a}{}^a(G) = \frac{D-3}{D-2}\kappa^2\rho \qquad \text{with} \qquad \rho = m\Phi^*\Phi\,, \tag{48}$$

$$R_{\mu b}{}^b{}_a(J) = 0\,. \tag{49}$$

For $D = 4$ the first equation, after gauge-fixing to a frame with constant acceleration, leads to the expected Newton's law

$$\triangle\Phi = 4\pi G_N\,\rho \tag{50}$$

but for $D = 3$ the source term vanishes.

Surprisingly, it turns out that there is a different way of taking the non-relativistic limit that only works in $D = 3$ dimensions. The possibility of this limit is due to the fact that for $D = 3$ the Bargmann algebra allows for a second central extension[12]:

$$\text{Galilei} \quad \overset{\text{‘Mass’}}{\Longrightarrow} \quad \text{Bargmann} \quad \overset{\text{‘Spin’}}{\Longrightarrow} \quad \text{Extended Bargmann}\,.$$

For a further discussion of this second central charge, see[13–15]. We will call the Bargmann Algebra with this second central charge the Extended Bargmann Algebra. Applying a gauging procedure on this Extended Bargmann algebra will lead to the so-called Extended Bargmann Gravity (EBG) model. We will first construct this EBG model and after that show how it can be obtained by defining a special limit of 3D General Relativity, augmented with *two* Abelian gauge fields.

The space-time symmetry algebra of EBG consists of the generators of time translations H, spatial translations P_a (with $a = 1, 2$), spatial rotations J, Galilean boosts G_a, a central charge M, corresponding to particle mass as well as a second central charge S[12–15]. The generators H, P_a, J, G_a and M form the Bargmann algebra and the inclusion of S leads to the Extended Bargmann Algebra whose non-zero commutation relations are given by

$$\begin{aligned} [H, G_a] &= -\epsilon_{ab} P_b\,, \qquad [J, G_a] = -\epsilon_{ab} G_b\,, \\ [J, P_a] &= -\epsilon_{ab} P_b\,, \quad [G_a, G_b] = \epsilon_{ab} S\,, \\ [G_a, P_b] &= \epsilon_{ab} M\,. \end{aligned} \tag{51}$$

Unlike the Bargmann algebra, the extended Bargmann algebra can be equipped with a non-degenerate, invariant bilinear form or ‘trace’, given by[16]

$$< G_a, P_b >= \delta_{ab}\,, \quad < H, S >=< M, J >= -1\,. \tag{52}$$

The action of EBG is given by the Chern-Simons action for the gauge algebra (51)

$$S = \frac{k}{4\pi} \int < A \wedge dA + \frac{2}{3} A \wedge A \wedge A >\,, \tag{53}$$

where k is the Chern-Simons coupling constant and the gauge field $A = A_\mu \mathrm{d}x^\mu$ is given by

$$A_\mu = \tau_\mu\, H + e_\mu{}^a\, P_a + \omega_\mu\, J + \omega_\mu{}^a\, G_a + m_\mu\, M + s_\mu\, S\,. \tag{54}$$

Explicitly, one finds the following action for EBG[16][c]

$$\begin{aligned} S = \frac{k}{4\pi} \int \mathrm{d}^3x \Big(& \epsilon^{\mu\nu\rho} e_\mu{}^a R_{\nu\rho}(G_a) - \epsilon^{\mu\nu\rho} m_\mu R_{\nu\rho}(J) \\ & - \epsilon^{\mu\nu\rho} \tau_\mu R_{\nu\rho}(S) \Big)\,, \end{aligned} \tag{55}$$

[c]We thank Diego Hofman for sharing with us the observation that a particular truncation of this action is related to the lower-spin gravity action introduced in[17].

where here and in the following we have used the curvatures

$$
\begin{aligned}
R_{\mu\nu}(H) &= 2\partial_{[\mu}\tau_{\nu]}\,,\\
R_{\mu\nu}(P^a) &= 2\partial_{[\mu}e_{\nu]}{}^a + 2\epsilon^{ab}\omega_{[\mu}e_{\nu]b} - 2\epsilon^{ab}\omega_{[\mu b}\tau_{\nu]}\,,\\
R_{\mu\nu}(J) &= 2\partial_{[\mu}\omega_{\nu]}\,,\\
R_{\mu\nu}(G^a) &= 2\partial_{[\mu}\omega_{\nu]}{}^a + 2\epsilon^{ab}\omega_{[\mu}\omega_{\nu]b}\,,\\
R_{\mu\nu}(M) &= 2\partial_{[\mu}m_{\nu]} + 2\epsilon^{ab}\omega_{[\mu a}e_{\nu]b}\,,\\
R_{\mu\nu}(S) &= 2\partial_{[\mu}s_{\nu]} + \epsilon^{ab}\omega_{[\mu a}\omega_{\nu]b}\,.
\end{aligned}
\tag{56}
$$

These curvatures are covariant with respect to the local H, P_a, J, G_a, M and S transformations of τ_μ, $e_\mu{}^a$, m_μ, ω_μ, $\omega_\mu{}^a$ and s_μ, that are found from the gauge algebra (51) following the usual rules of gauge theory. Note that the fields τ_μ, $e_\mu{}^a$, ω_μ, $\omega_\mu{}^a$ and m_μ also appear in the formulation of NC gravity obtained by gauging the Bargmann algebra. As in that case, τ_μ and $e_\mu{}^a$ can be interpreted as Vielbeine for two degenerate time-like and spatial metrics, respectively. The field s_μ is not present in NC gravity and is specific to EBG. We note that the equations of motion for s_μ, ω_μ and $\omega_\mu{}^a$ lead to the curvature constraints

$$
R_{\mu\nu}(H) = 0\,, \quad R_{\mu\nu}(P^a) = 0\,, \quad R_{\mu\nu}(Z) = 0\,,
\tag{57}
$$

that are usually imposed by hand in NC gravity. As in NC gravity, these equations imply that EBG is defined on non-relativistic space-times with torsionless NC geometry. The first equation implies that the space-time can be foliated in an absolute time direction, while the last two equations can be used to express ω_μ and $\omega_\mu{}^a$ in terms of τ_μ, $e_\mu{}^a$ and m_μ. Following NC gravity, ω_μ and $\omega_\mu{}^a$ can be seen as appropriate non-relativistic versions of the relativistic spin connection.

Having constructed the EBG model, we will now show how the EBG action (55) can be obtained as the non-relativistic limit of a suitable extension of the three-dimensional Einstein-Hilbert action. In order to show this, we extend the procedure that allows one to obtain the equations of motion of NC gravity from Einstein's equations, that was discussed in the previous section. As a starting point, we take the following Einstein-Hilbert action for the relativistic Vielbein $E_\mu{}^A$ and spin connection $\Omega_\mu{}^{AB}$, written as a Chern-Simons action, plus a Chern-Simons action for two Abelian gauge fields $Z_{1\mu}$ and $Z_{2\mu}$:

$$
S = \frac{k\omega}{4\pi}\int \mathrm{d}^3x\left(\epsilon^{\mu\nu\rho}\,E_\mu{}^A\,R_{\nu\rho}(J_A) + 2\,\epsilon^{\mu\nu\rho}\,Z_{1\mu}\,\partial_\nu Z_{2\rho}\right),
\tag{58}
$$

where the Riemann tensor $R_{\mu\nu}(J^A)$ reads

$$
R_{\mu\nu}(J^A) = 2\partial_{[\mu}\Omega_{\nu]}{}^A - \epsilon^{ABC}\Omega_{[\mu B}\Omega_{\nu]C}\,.
\tag{59}
$$

Extending the particle-limit procedure of [6], mimicking the Inönü-Wigner contraction of the underlying Poincaré $\otimes$ U$(1)^2$ gauge algebra, we express the relativistic gauge fields $E_\mu{}^A$, $\Omega_\mu{}^A$, $Z_{1\mu}$, $Z_{2\mu}$ in terms of the non-relativistic fields

$\tau_\mu, e_\mu{}^a, \omega_\mu{}^a, \omega_\mu, m_\mu, s_\mu$ as follows:

$$
\begin{aligned}
E_\mu{}^0 &= \omega\,\tau_\mu + \frac{1}{2\omega}\,m_\mu\,, & Z_{1\mu} &= \omega\,\tau_\mu - \frac{1}{2\omega}\,m_\mu\,,\\
\Omega_\mu{}^0 &= \omega_\mu + \frac{1}{2\omega^2}\,s_\mu\,, & Z_{2\mu} &= \omega_\mu - \frac{1}{2\omega^2}\,s_\mu\,,\\
E_\mu{}^a &= e_\mu{}^a\,, & \Omega_\mu{}^a &= \frac{1}{\omega}\,\omega_\mu{}^a\,.
\end{aligned}
\tag{60}
$$

Using these expressions in the action (58) and taking the limit $\omega \to \infty$ [d] it is straightforward to show that the EBG action (55) is obtained.

In the next section we will show that EBG and NC gravity are different theories by comparing their coupling to matter.

6. Matter coupled Extended Bargman Gravity

Introducing matter couplings in EBG can be done by adding one of the matter Lagrangians on arbitrary torsionless NC backgrounds constructed in[5]:

$$
\begin{aligned}
S = \frac{k}{4\pi}\int \mathrm{d}^3x\,\Big(&\epsilon^{\mu\nu\rho}e_\mu{}^aR_{\nu\rho}(G_a) - \epsilon^{\mu\nu\rho}\tau_\mu R_{\nu\rho}(S)\\
&- \epsilon^{\mu\nu\rho}m_\mu R_{\nu\rho}(J)\Big) + \int \mathrm{d}^3x\,e\,\mathcal{L}_{\mathrm{m}}\,.
\end{aligned}
\tag{61}
$$

Here $e = \det(\tau_\mu, e_\mu{}^a)$ denotes the volume element. Since any matter couplings to the s_μ gauge field change the foliation constraint $R_{\mu\nu}(H) = 0$, we do not consider such couplings so that we can stay within the framework of torsionless NC geometry.

Defining the energy current t^μ, the momentum current $t^\mu{}_a$ and the particle number current j^μ by

$$
\begin{aligned}
t^\mu &= \frac{1}{e}\frac{\delta}{\delta\tau_\mu}\left(e\mathcal{L}_{\mathrm{m}}\right)\,, \qquad t^\mu{}_a = \frac{1}{e}\frac{\delta}{\delta e_\mu{}^a}\left(e\mathcal{L}_{\mathrm{m}}\right)\,,\\
j^\mu &= \frac{1}{e}\frac{\delta}{\delta m_\mu}\left(e\mathcal{L}_{\mathrm{m}}\right)\,,
\end{aligned}
\tag{62}
$$

the equations of motion stemming from the action (61) take the form

$$
\begin{aligned}
e^{-1}\epsilon^{\mu\nu\rho}R_{\nu\rho}(S) &= \frac{4\pi}{k}t^\mu\,, \quad e^{-1}\epsilon^{\mu\nu\rho}R_{\nu\rho}(J) = \frac{4\pi}{k}j^\mu\,,\\
e^{-1}\epsilon^{\mu\nu\rho}R_{\nu\rho}(G_a) &= -\frac{4\pi}{k}t^\mu{}_a\,.
\end{aligned}
\tag{63}
$$

Since the curvatures in these equations obey Bianchi identities, the currents obey various identities for consistency. We distinguish between Bianchi identities ‘of the

[d] This limit is well-defined. The term $-2\,\epsilon^{\mu\nu\rho}E_\mu{}^0\,\partial_\nu\Omega_\rho{}^0$ leads to a potentially diverging $-2\,\omega^2\,\epsilon^{\mu\nu\rho}\tau_\mu\,\partial_\nu\omega_\rho$ term, but this term gets cancelled by a contribution coming from the term $2\,\epsilon^{\mu\nu\rho}\,Z_{1\mu}\,\partial_\nu Z_{2\rho}$ that we added to the Einstein-Hilbert action.

first kind' and 'of the second kind'. The identities of the first kind follow from the fact that the equations

$$R_{\mu\nu}(P^a) = 0\,, \qquad R_{\mu\nu}(Z) = 0 \tag{64}$$

are identically satisfied, once one views the spin connections ω_μ and $\omega_\mu{}^a$ as dependent on τ_μ, $e_\mu{}^a$ and m_μ. Substituting equations (64) into the Bianchi identities $D_{[\mu}R_{\nu\rho]}(P^a) = 0$ and $D_{[\mu}R_{\nu\rho]}(Z) = 0$ leads to the following Bianchi identities of the first kind:

$$R_{[\mu\nu}(J)\, e_{\rho]}{}^a = R_{[\mu\nu}(G^a)\, \tau_{\rho]}\,, \qquad \epsilon^{ab} R_{[\mu\nu}(G_a)\, e_{\rho]b} = 0\,. \tag{65}$$

The remaining Bianchi identities, called of the second kind, are not algebraic in the curvatures and are given by

$$D_{[\mu}R_{\nu\rho]}(G^a) = 0\,, \quad \partial_{[\mu}R_{\nu\rho]}(J) = 0\,, \quad D_{[\mu}R_{\nu\rho]}(S) = 0\,. \tag{66}$$

Combining the equations of motion (63) with the Bianchi identities of the first kind (65) leads to the following algebraic consistency conditions

$$e_\mu{}^a j^\mu = -\tau_\mu t^\mu{}_a\,, \qquad e_\mu{}^{[a} t^{|\mu|b]} = 0\,. \tag{67}$$

The Bianchi identities of the second kind on the other hand lead to the following current conservation conditions:

$$D_\mu t^\mu = 0\,, \qquad D_\mu t^\mu{}_a = 0\,, \qquad D_\mu j^\mu = 0\,. \tag{68}$$

The EBG equations of motion (63) are strikingly different from the NC gravity ones. Using the following identity between the Riemann tensor and the curvatures $R(J)$ and $R(G)$

$$R^\sigma{}_{\rho\mu\nu} = \epsilon^{ab} R_{\mu\nu}(J) e^\sigma{}_a e_{\rho b} - \epsilon^{ab} R_{\mu\nu}(G_b) e^\sigma{}_a \tau_\rho \tag{69}$$

it follows from the equations of motion (63) that the purely time-like component of the Ricci tensor $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ is given by $\tau^\mu \tau^\nu R_{\mu\nu} \propto e_\mu{}^a t^\mu{}_a$. This is unlike NC gravity, where one rather has $\tau^\mu \tau^\nu R_{\mu\nu} \propto j^0$ [e]. Furthermore, in NC gravity only this purely time-like component of the Ricci tensor is non-zero. This is in contrast with EBG where matter sources all components of the Riemann tensor. As a result, three-dimensional EBG admits backgrounds with non-trivial curvature whenever matter is present.

[e]It is instructive to consider this difference in the example of a massive complex Schrödinger field Φ, whose action can be found in [5]. In that case one finds for EBG that $\tau^\mu\tau^\nu R_{\mu\nu} = -\frac{2\pi \mathrm{i}}{k}\left(\Phi^* D_0 \Phi - \Phi D_0 \Phi^*\right)$ while the case of NC gravity leads to $\tau^\mu\tau^\nu R_{\mu\nu} \propto |\Phi|^2$.

7. Future Directions

We have shown that in three spacetime dimensions Extended Bargmann Gravity (EBG), a non-relativistic gravity theory that is based upon a Galilei algebra with two central extensions, can be obtained as a non-relativistic limit of a suitable generalization of the Einstein-Hilbert action. This EBG model has different matter couplings than NC gravity and allows, unlike NC gravity, background solutions with *curved* space.[f] Interestingly, a supergravity version of EBG can be constructed, as was shown in[24]. This new Extended Bargmann Supergravity opens up the possibility of obtaining exact non-perturbative quantities of non-relativistic supersymmetric field theories. Indeed, using the techniques of[20] one can use these results to construct supersymmetric quantum field theories in a non-trivial curved background. Following[21,22] one can then apply localization techniques to extract exact results out of such theories. The first step in this program for the case of NC gravity has already been taken in[23].

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